Exact phase diagram of a model with aggregation and chipping

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We reexamine a simple lattice model of aggregation in which masses diffuse and coalesce upon contact with rate 1 and every nonzero mass chips off a single unit of mass and adds it to a randomly chosen neighbor with rate w. The dynamics conserves the average mass density ρ and in the stationary state the system undergoes a nonequilibrium phase transition in the (ρ-w) plane across a critical line $\rho_c(w)$. In this paper, we show analytically that in arbitrary spatial dimensions $\rho_c(w) = \sqrt{w+1} - 1$ exactly and hence, remarkably, is independent of dimension. We also provide direct and indirect numerical evidence that strongly suggests that the mean field asymptotic results for the single site mass distribution function and the associated critical exponents are superuniversal, i.e., independent of dimension.

I. INTRODUCTION

Nonequilibrium phase transitions [1] occur in various systems including heterogeneous catalysis [2], chemical reaction models [3], polynuclear growth models [4], monomer-dimer models [5], models of fungal growth [6], nonequilibrium kinetic Ising models [7], and branching annihilating random walks [8]. A common feature in all the above is that the transition is from a state that has no activity to one that has continued activity. Such transitions are well characterized by the critical exponents of directed percolation (DP), parity conserving, or DP2 universality classes. There are other nonequilibrium models that undergo phase transitions which do not belong to the above universality classes. These include boundary driven phase transitions [9] and models whose steady states undergo a depinning or unbinding transition [10–13]. Recently, a simple lattice model where masses diffuse, aggregate on contact, and also chip off a single unit of mass was studied [14,15]. This chipping model (CM) exhibits a nonequilibrium phase transition [14,15] in its steady state from a phase in which an *infinite* aggregate is present to one that has none. This nonequilibrium phase transition is in a completely different universality class from the other models studied in the literature and mentioned above. The mathematical mechanism giving rise to the formation of the infinite aggregate at the onset of the phase transition was found to be very similar to that of the equilibrium Bose-Einstein condensation in an ideal Bose gas. The difference is that in the CM the infinite aggregate or condensate forms in real space as opposed to the Bose gas where the condensation takes place in momentum space. In addition, the phase transition in the CM occurs even in one dimension as opposed to the Bose gas where the condensation occurs in two and higher dimensions. A slightly different off-lattice version of the CM was studied earlier within the rate equation approach in the context of aggregation in dry environments [16]. A directed version of the CM where masses move asymmetrically only along one direction was studied in Ref. [15] and its critical properties were found to belong to a different universality class from that of the undirected CM. This directed CM also appeared recently in the context of a traffic model with passing [17]. Related models of fragmentation and diffusion have been studied in [18–20].

The undirected CM is defined on a d-dimensional hypercubic lattice with periodic boundary conditions. Starting from a random distribution of non-negative integer masses at each lattice site, the system evolves in continuous time via the following microscopic processes. In an infinitesimal time interval Δt, (i) with probability Δt, the mass at each site hops to one of its neighboring sites, chosen at random, (ii) with probability $w\Delta t$, a unit mass is chipped off from an already existing mass at each site and added to one of the neighboring sites, again chosen at random, and (iii) with probability $[1 - (1+w)\Delta t]$, each mass stays at its original site. Following the steps (i)–(iii), the masses at any given site are added up. Since this system is closed to the environment (periodic boundary condition), the total mass is conserved by the dynamics. Thus, there are two parameters in the problem, the average mass per site ρ, and the ratio of the chipping rate to the rate of hopping as a whole, w. In the long time limit, the system evolves into a time independent steady state. The steady state single site mass distribution function $P(m)$, i.e., the probability that a site has mass m when $t \to \infty$, was shown to undergo a phase transition in the ρ-w plane [14,15]. There is a critical line $\rho_c(w)$ in the ρ-w plane that separates two types of asymptotic behavior of $P(m)$. For fixed w, as ρ is varied across the critical value $\rho_c(w)$, the large m behavior of $P(m)$ was found to be

$$P(m) \sim \begin{cases} e^{-m/m^*}, & \rho < \rho_c(w) \\ m^{-\tau}, & \rho = \rho_c(w) \\ m^{-\tau} + \text{infinite aggregate}, & \rho > \rho_c(w). \end{cases} \quad (1)$$

Thus, the tail of the mass distribution changes from an exponential decay to an algebraic decay as ρ approaches ρ_c from below. As one increases ρ beyond ρ_c, this asymptotic algebraic part of the critical distribution remains unchanged but in addition an infinite aggregate forms. This means that all the additional mass $(\rho - \rho_c)V$ (where V is the volume of the system) condenses onto a single site and does not disturb the background critical distribution. This is analogous, in

spirit, to the condensation of a macroscopic number of bosons onto the single $k=0$ mode in an ideal Bose gas as the temperature goes below a certain critical value. This infinite tower, i.e., the single site carrying macroscopically large (proportional to volume) mass, is mobile. Mathematically this means that for $\rho > \rho_c(w)$ $P(m) \sim m^{-\tau} + (1/V)\delta(m - \alpha V)$ where $\alpha = \rho - \rho_c$. This last δ-function peak contributes only to order $1/V$ in the integral $\int P(m)dm$, indicating that the aggregate is contained in thermodynamically few sites (indeed, a single site) but contributes a finite amount α in the integral $\int mP(m)dm$ even in the thermodynamic limit $V \to \infty$.

These results were found both analytically within a mean field approximation, which involved ignoring all correlations between masses, and numerically in one dimension. Within the mean field approximation, the locus of the critical line was found [14] to be $\rho_c(w) = \sqrt{w+1} - 1$ and the exponent $\tau = 5/2$ [14,16]. In one dimension, the numerically obtained critical line [14] was found to be close to the mean field phase curve. Even the exponent τ, determined from a simple linear fit on the log-log plot of data from relatively smaller size lattices, was found to be $\tau \approx 2.33$, rather close to the mean field exponent 2.5. This raises the question whether or not the mean field results for asymptotic behaviors of $P(m)$ are exact even in one dimension. On the other hand, one can show explicitly (see Sec. II) that there exist nonzero correlations between masses in any finite dimension, even in the thermodynamic limit. Thus, one is confronted with a puzzle.

The purpose of this paper is to shed light on this puzzling issue. We first prove analytically the remarkable result that the mean field phase boundary $\rho_c(w) = \sqrt{w+1} - 1$ is indeed *exact* and independent of the spatial dimension d. This, of course, still does not prove, but hints, that the exponent τ may also be independent of d. However, we provide rather unambiguous numerical evidence, in conjunction with several direct and indirect checks, which suggests that even the exponent $\tau = 5/2$ is superuniversal and independent of d. Thus, our results seem to suggest strongly that, even though there are nonzero correlations between masses in finite dimensions, these correlations do not affect the asymptotic behavior of the single site steady state mass distribution $P(m)$. However, the possibility remains that other higher order correlation functions, such as the joint distribution of two masses $P(m_1,m_2)$, will depend on the spatial dimension d.

The paper is organized as follows. In Sec. II, we show analytically that in arbitrary dimension d the locus of the phase boundary is independent of d and is given by the mean field expression $\rho_c(w) = \sqrt{w+1} - 1$. In Sec. III, we rederive analytically the mean field expression for the mass distribution $P(m)$ by a method different from that used in [14] and compare these analytical expressions with the numerical results obtained in one and two dimensions. In Sec. IV, we do a finite size scaling analysis that provides additional evidence that the exponent τ is superuniversal. Finally, we conclude with a summary and discussion in Sec. V.

II. EXACT PHASE DIAGRAM
IN ARBITRARY DIMENSIONS

As mentioned in the Introduction, the steady state of the CM undergoes a phase transition in its parameter space (ρ-w

plane) across a critical line $\rho_c(w)$ in all dimensions. In this section, we compute $\rho_c(w)$ exactly in arbitrary dimensions by analyzing the two-point equal time mass-mass correlation function $C(\mathbf{x},t) = \langle m(\mathbf{x},t)m(\mathbf{0},t)\rangle$. Let $\eta(\mathbf{x},\mathbf{x}',t)$ denote the mass transferred from a site $\mathbf{x}$ to a neighboring site $\mathbf{x}'$ in the time interval between t and $t+\Delta t$. Clearly, $\eta(\mathbf{x},\mathbf{x}',t)$ is a random variable that takes the values

$$\eta(\mathbf{x},\mathbf{x}',t)$$

$$= \begin{cases} m(\mathbf{x}) & \text{with probability} & (1/2d)\Delta t \\ 1 - \delta_{m(\mathbf{x}),0} & \text{with probability} & (1/2d)w\Delta t \\ 0 & \text{with probability} & 1 - [(1+w)/2d]\Delta t, \end{cases}$$

$$(2)$$

where $2d$ is the number of neighbors of any given site. The Kronecker δ function in the second line of Eq. (2) indicates that a chipping of a unit mass can take place provided the mass $m(\mathbf{x})$ is a positive integer bigger than 0. Then, the evolution of mass at site $\mathbf{x}$ can be written as

$$m(\mathbf{x},t+\Delta t) = m(\mathbf{x},t) - \sum_{\mathbf{x}'} \eta(\mathbf{x},\mathbf{x}',t) + \sum_{\mathbf{x}'} \eta(\mathbf{x}',\mathbf{x},t),$$

$$(3)$$

where the sum is over the neighbors $\mathbf{x}'$ of the site $\mathbf{x}$. The second term on the right hand side of Eq. (3) describes the outflow of mass from $\mathbf{x}$ while the third term accounts for the inflow of mass into $\mathbf{x}$ from neighboring sites. It is quite straightforward to write down the two-point correlator for η to order Δt. Suppressing the explicit t dependence in η, we find

$$\langle \eta(\mathbf{x}_1,\mathbf{x}_1')\eta(\mathbf{x}_2,\mathbf{x}_2')\rangle = \frac{1}{2d}[m^2(\mathbf{x}_1) + w(1 - \delta_{m(\mathbf{x}_1),0})]$$

$$\times \delta_{\mathbf{x}_1,\mathbf{x}_2}\delta_{\mathbf{x}_1',\mathbf{x}_2'}\Delta t. \quad (4)$$

Using Eqs. (3) and (4), the evolution equations for the two-point equal time correlation function $C(\mathbf{x},t) = \langle m(\mathbf{x},t)m(\mathbf{0},t)\rangle$ can be written down. Multiplying Eq. (3) by $m(\mathbf{0},t+dt)$ and taking the average on both sides, and putting all time derivatives to zero in the steady state, we get,

$$-C(\mathbf{x}) + \frac{1}{2d}\sum_{\mathbf{x}'} C(\mathbf{x}')$$

$$= w\left(-D(\mathbf{x}) + \frac{1}{2d}\sum_{\mathbf{x}'} D(\mathbf{x}')\right)$$

$$-[C(\mathbf{0}) + ws]\left(\delta_{\mathbf{x},0} - \frac{1}{2d}\sum_{\mathbf{x}_0} \delta_{\mathbf{x},\mathbf{x}_0}\right), \quad (5)$$

where $\mathbf{x}_0$ denotes the neighbors of the site $\mathbf{0}$. Also, $D(\mathbf{x},t) = \langle m(\mathbf{x},t)\delta_{m(\mathbf{0},t),0}\rangle$, and $s = 1 - \langle \delta_{m(\mathbf{0}),0}\rangle$ is the probability of a site having nonzero mass. Thus, in the CM, the two-point correlation functions do not form a closed set of equations, making it difficult to solve for $C(\mathbf{x},t)$ exactly. This is unlike many other models where two-point correlations do form a

closed set of equations and hence are solvable. A few examples include the one-dimensional (1D) Glauber model [21], asymmetric random average processes [19,20], the Takayasu model of aggregation [22], and the q model of force fluctuations in bead packs [22,23].

Remarkably, however, Eq. (5) allows for the solution

$$C(\mathbf{x}) = w[D(\mathbf{x}) - s], \quad \mathbf{x} \neq \mathbf{0}. \tag{6}$$

This solution is also the unique solution. To see this, observe that the homogeneous part of Eq. (5) is the Laplace equation $\nabla^2[C(\mathbf{x}) - wD(\mathbf{x})] = 0$, with the boundary condition that $C(\mathbf{x}) - wD(\mathbf{x})$ is a constant as $|\mathbf{x}| \to \infty$. Since the solution Eq. (6) satisfies the inhomogeneous part too, as well as the boundary conditions, it is the unique solution.

Note that the above solution Eq. (6) is also valid on a finite lattice. Summing Eq. (6) over all $\mathbf{x} \neq \mathbf{0}$ and using the fact that the conserved total mass is given by $\sum_{\mathbf{x}} m(\mathbf{x}) = \rho V$ (where $V = L^d$ is the volume of the system), we get the exact equation

$$\rho^2 - \frac{\langle m^2 \rangle}{V} = w\rho(1-s) - ws. \tag{7}$$

This equation is reminiscent of Bose-Einstein condensation in an ideal Bose gas. In the low density phase, we expect that the system reaches a stationary state in which $\langle m^2 \rangle$ is a finite number of order $O(1)$. Therefore, in the thermodynamic limit $V \to \infty$, the second term on the left hand side of Eq. (7) drops out and we get

$$\rho^2 = w\rho(1-s) - ws. \tag{8}$$

Note that Eq. (8) could have been obtained from Eq. (6) if we had assumed that the two-point correlation functions decouple, i.e., $C(\mathbf{x}) = \langle m(\mathbf{0})m(\mathbf{x}) \rangle = \langle m(\mathbf{0}) \rangle \langle m(\mathbf{x}) \rangle = \rho^2$ and $D(\mathbf{x}) = \langle m(\mathbf{x}) \delta_{m(\mathbf{0}),0} \rangle = \rho(1-s)$. However, there is no reason *a priori* for the two-point correlations to decouple. Our derivation of Eq. (8) does not rely on this decoupling.

From Eq. (8) we get

$$s = \frac{w\rho - \rho^2}{w(\rho+1)}. \tag{9}$$

According to Eq. (9), as one increases the density ρ, keeping w fixed, the occupation probability s first increases with ρ, attains a maximum at $\rho = \sqrt{w+1} - 1$ [obtained by setting $ds/d\rho = 0$ in Eq. (9)], and then starts decreasing with increasing ρ. However, it is clear that s, the probability that a site has nonzero mass, must be a monotonically nondecreasing function of ρ. Hence we conclude that Eq. (9) is valid as long as $\rho \leq \rho_c = \sqrt{w+1} - 1$. For $\rho > \rho_c$, the basic assumption $\langle m^2 \rangle / V \to 0$ as $V \to \infty$ breaks down and Eq. (9) ceases to be valid.

Thus, the critical density is given by

$$\rho_c(w) = \sqrt{1+w} - 1, \tag{10}$$

and, remarkably, it is independent of d and not surprisingly, therefore, coincides with the mean field expression [14,15].

For $\rho \leq \rho_c$, s is given by Eq. (9). As ρ increases from 0 to $\rho_c(w)$ (for fixed w), s increases monotonically according to Eq. (9) up to the value s_c given by

$$s_c = \frac{\sqrt{1+w} - 1}{\sqrt{1+w} + 1}. \tag{11}$$

For $\rho > \rho_c(w)$, s does not increase any further and sticks to its value s_c. Putting $s = s_c$ in Eq. (7) and using the expression of s_c from Eq. (11), we get for $\rho > \rho_c(w)$

$$\lim_{V \to \infty} \left\langle \frac{m^2}{V} \right\rangle = (\rho - \rho_c)^2. \tag{12}$$

Thus, for $\rho > \rho_c$, $\langle m^2 \rangle$ becomes macroscopic, i.e., proportional to volume. Since s, the fraction of occupied sites, does not increase any longer for $\rho > \rho_c$, this indicates that all the extra mass $(\rho - \rho_c)V$ condenses onto a thermodynamically negligible number of sites (indeed, a single site only) with density $\sim 1/V$, leading to the macroscopic behavior of $\langle m^2 \rangle \sim (1/V)[(\rho - \rho_c)V]^2 \sim (\rho - \rho_c)^2 V$. This is similar in spirit, though not in detail, to the Bose-Einstein condensation, where below a certain temperature the number of particles in the $k = 0$ mode also becomes macroscopic.

Let us conclude this section by stressing an important point. We note that the mean field solution (assuming decoupling) for the stationary two-point correlation function, $C(\mathbf{x}) = \rho^2$ for $\mathbf{x} \neq \mathbf{0}$ and $D(\mathbf{x}) = \rho(1-s)$ for $\mathbf{x} \neq \mathbf{0}$ with s satisfying Eq. (7), is indeed an exact solution of Eqs. (5) and (6). However, this need not be the only stationary solution. In addition even if the mean field result for the two-point stationary correlation function is the correct one, it still does not prove that the mean field theory is exact. For example, one can show that indeed three- and higher point correlation functions do not decouple. In any case, the main result of this section, namely, the derivation of the exact phase boundary, does not rely on whether the correlation functions decouple or not.

III. COMPARISON WITH MEAN FIELD THEORY

In the previous section, we proved that the mean field phase diagram is exact in any dimension. This, of course, does not prove, but suggests that, perhaps, even the mean field expression for the distribution $P(m)$ may also be asymptotically exact in all dimensions. In this section, we try to provide evidence in favor of this hypothesis. For this we first rederive the mean field expression for $P(m)$ and compare it with the numerical results obtained in one and two dimensions.

In Ref. [14], the steady state single site mass distribution function $P(m)$ was computed analytically by assuming that the joint distribution $P(m_1, m_2)$, the probability that two consecutive sites have masses m_1 and m_2, respectively, factorizes, i.e., $P(m_1, m_2) = P(m_1)P(m_2)$. With this assumption, $P(m)$ was shown to satisfy a closed set of equations which were then solved via the generating function method, yielding the results given in Eq. (1) with $\tau = 5/2$ and $\rho_c(w)$

$=\sqrt{w+1}-1$. In this section, we first rederive the mean field results by a different method that requires fewer restrictions than the product measure used in Ref. [14].

Here we use a technique used before for solving the mass distribution function in other models of aggregation [24] as well as in the $w=0$ limit of the CM [25]. We consider the CM on a 1D lattice. Let $P(m_1,m_2,\ldots,m_n)$ denote the joint probability that n consecutive sites on the lattice have masses m_1, m_2, $\ldots$, m_n, respectively, in the stationary state. We define two generating functions,

$$Z_n = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} x^{m_1+\cdots+m_n} P(m_1,\ldots,m_n), \quad (13)$$

$$Y_n = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} x^{m_1+\cdots+m_n} P(m_1,\ldots,m_n,0). \quad (14)$$

Here $Z_n = \langle x^{m_1+\cdots+m_n}\rangle$ is an unconditional average but $Y_n = \langle x^{m_1+\cdots+m_n}\rangle_0$ is a conditional average where the $(n+1)$th site is conditioned to have 0 mass. Using the dynamics of m_i [as given by Eq. (3)] and following steps similar to those used in Refs. [24,25], one can write down the evolution equations for Z_n. In the steady state, when all time derivatives go to zero, we get, after some algebra,

$$Z_{n+1} - 2Z_n + Z_{n-1} + w\left(\frac{(1-x)^2}{x}Z_n\right.$$

$$\left. - \frac{1-x}{x}Y_{n-1} + (1-x)Y_n\right) = 0, \quad (15)$$

with the boundary conditions $Z_0=1$ and $Y_0=P(0)=1-s$, the probability of having no mass at any given site.

If we now make the assumption that $P(m_1+\cdots+m_n=m,0)=P(m_1+\cdots+m_n=m)P(0)$, i.e., $Y_n=P(0)Z_n$, then we get equations that contain only the Z_n's. As mentioned before, this assumption is less strict than the product measure approximation as it requires the factorization of only a special conditional probability where a site at the beginning of a string is empty. In order to determine $P(m)$, we need to compute $Z_1(x)$ which, by definition in Eq. (14), is the generating function for the $P(m)$'s,

$$Z_1(x) = \sum_{m=0}^{\infty} x^m P(m). \quad (16)$$

Since Z_1 depends on other Z_n's, we need to solve the full Eq. (15) for all n. Equation (15) can be solved by the standard generating function method. Let $G(x,y)=\sum_{n=1}^{\infty}Z_n(x)y^n$. Multiplying Eq. (15) by y^n and summing over n, we get, after straightforward algebra,

$$G(x,y) = \frac{y[wP(0)y(1-x)-xy+xZ_1(x)]}{x(1-y)^2 + wy(1-x)^2 + wP(0)y(1-x)(x-y)}. \quad (17)$$

For a fixed x, when considered as a function of y only, $G(x,y)$ has two poles,

$$y_{1,2} = \frac{-2x + w(1-x)(1-sx) \pm (1-x)\sqrt{w}\sqrt{w(1-sx)^2-4sx}}{2((1-s)w(1-x)-x)}. \quad (18)$$

For a fixed w and s, $|y_2|<1$ for small values of x. This implies that $Z_n \sim |y_2|^{-n}$ for large n. However, we cannot have a diverging probability for large n. Hence this pole must be canceled by the numerator of G. This pole canceling mechanism was also useful in deriving exact results in other recently studied aggregation models [20,22]. Demanding that $G(x,y_2)=0$, we get the following expression for $Z_1(x)$:

$$Z_1(x) = \frac{2x - w(1-x)(1-sx) + (1-x)\sqrt{w}\sqrt{w(1-sx)^2-4sx}}{2x}. \quad (19)$$

The coefficient of x^m on the right hand side of Eq. (19) will then give the desired distribution $P(m)$.

The expression for the generating function $Z_1(x) = \sum_{m=0}^{\infty} x^m P(m)$ in Eq. (19) is identical to the one derived in [14] using the approximation of the product measure. Thus, the two methods, although different in detail, yield the same $P(m)$ for *any* m and not just for large m. This result for the mean field $P(m)$ seems to be extremely robust and does not depend on the details of how the mean field assumption is incorporated.

The asymptotic properties of $P(m)$ for large m can be derived by analyzing the behavior of $Z_1(x)$ near $x=1$ [14],

and one recovers the results in Eq. (1) with $\tau=5/2$ and $\rho_c(w)=\sqrt{w+1}-1$. However, by expanding the expression for $Z_1(x)$ in Eq. (19) in powers of x using MATHEMATICA, we have determined $P(m)$ for all m. In the aggregate phase ($\rho>\rho_c$), we set $s=s_c$ in Eq. (19) and calculate the distribution $P(m)$ by expanding in powers of x. In Fig. 1, we compare this analytical mean field answer for all m with the numerical results obtained in one and two dimensions in the aggregate phase ($\rho>\rho_c$). Note that Fig. 1 shows only the power law part of the spectrum. The numerical data are for $V=900$ in one dimension and for a 30×30 lattice in two dimensions. The two curves for 1D and 2D are almost indis-
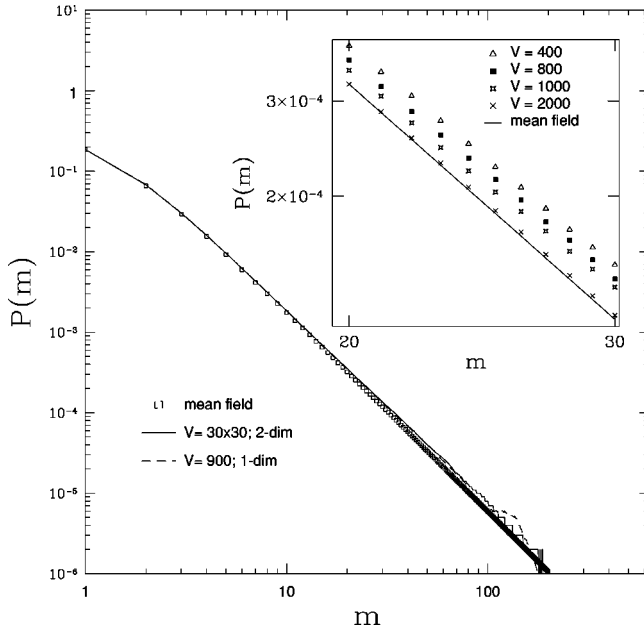
FIG. 1. The power law parts of the steady state mass distributions $P(m)$ in one and two dimensions are compared with the mean field result in the aggregate phase $\rho > \rho_c$. The data are for $w = 3$ and $\rho = 10$. The critical value is $\rho_c = 1$ for $w = 3$. In the inset panel, we show the convergence of the probability distribution to its mean field value as the system size is increased in one dimension.
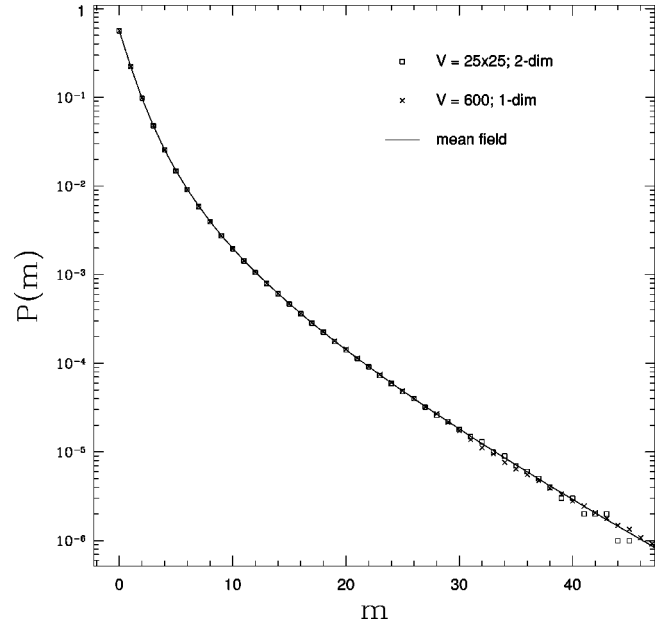

FIG. 2. The steady state mass distributions in one and two dimensions in the exponential phase ($\rho < \rho_c$) are compared with the mean field result. The data are for $w = 8$ and $\rho = 1$. The critical value is $\rho_c = 2$ for $w = 8$.

tinguishable from each other. While the numerical data match the mean field result excellently for small m, there is a small deviation for larger masses. This deviation at large m is due to finite size effects. To confirm this, we also did simulations for larger lattice sizes up to $V = 2000$ in one dimension. For a fixed mass, we confirmed that the deviation from the mean field decreases with increasing V (see the inset in Fig. 1). We also compared the mass distribution in the exponential phase ($\rho < \rho_c$) with the mean field prediction. Again, excellent agreement is seen (see Fig. 2). These results thus provide strong evidence that the mean field expression for $P(m)$ is exact in all dimensions and therefore the exponent $\tau = 5/2$ is also superuniversal, i.e., independent of d.

IV. FINITE SIZE SCALING AND INDIRECT NUMERICAL CHECKS

In this section, we provide further indirect numerical checks which again strongly suggest that the mean field result for $P(m)$ is indeed exact. We start by making a reasonable finite size scaling ansatz for $P(m,V)$ (where V is the volume of the system) in the aggregate phase $\rho > \rho_c$ for large m,

$$P(m,V) \approx \frac{1}{m^\tau} f\left(\frac{m}{V^\phi}\right) + \frac{1}{V} \delta(m - (\rho - \rho_c)V), \quad (20)$$

where the exponent ϕ is a crossover exponent and the δ function peaked at $(\rho - \rho_c)V$ indicates the aggregate containing a macroscopic amount of mass. The power law part of the mass distribution is cut off at $\sim V^\phi$ for finite V. Since the

largest mass inside the power law part is much smaller than that in the aggregate, we get the inequality $\phi < 1$.

Also, the mass density $\int m P(m) dm = \rho$ is finite. Substituting Eq. (20) in this integral, it is evident that for large V the integral will converge provided $\tau > 2$. Also, from Eq. (20), one finds that the second moment $\langle m^2 \rangle / V \approx (\rho - \rho_c)^2 + O(V^{\phi(3 - \tau) - 1})$. The first term is from the aggregate and the second term is from the power law part. This is consistent with the exact result Eq. (12) provided that $\phi(3 - \tau) < 1$. This provides an upper bound for τ, i.e., $\tau < 3$. Thus, we have the exact bounds for τ, $2 < \tau < 3$. The mean field results and numerical simulations in 1D and 2D are consistent with these bounds.

Next we derive a scaling relation between τ and ϕ. This is obtained by demanding that $P(m,V)$ is normalized, $\int P(m,V) dm = 1$. Substituting Eq. (20) in this integral, we get

$$\int_{m_l}^\infty \frac{1}{m^\tau} f\left(\frac{m}{V^\phi}\right) dm = c - \frac{1}{V}, \quad (21)$$

where c is a constant of $O(1)$, and $m_l \sim O(1)$ is the mass beyond which the scaling starts holding. Differentiating Eq. (21) twice with respect to V and making a change of variable, we arrive at the relation

$$\int_{m_l V^{-\phi}}^\infty f''(y) y^{2 - \tau} dy = -(1 - \phi) V^{\phi \tau - 1 - 1}, \quad (22)$$

where $f''(y) = d^2 f / d^2 y$. The reason for double differentiation is as follows. Since $2 < \tau < 3$, one can replace the lower limit of the integral in Eq. (22) by 0 in the $V \rightarrow \infty$ limit and there is no divergence from the lower cutoff. Once the lower cut-
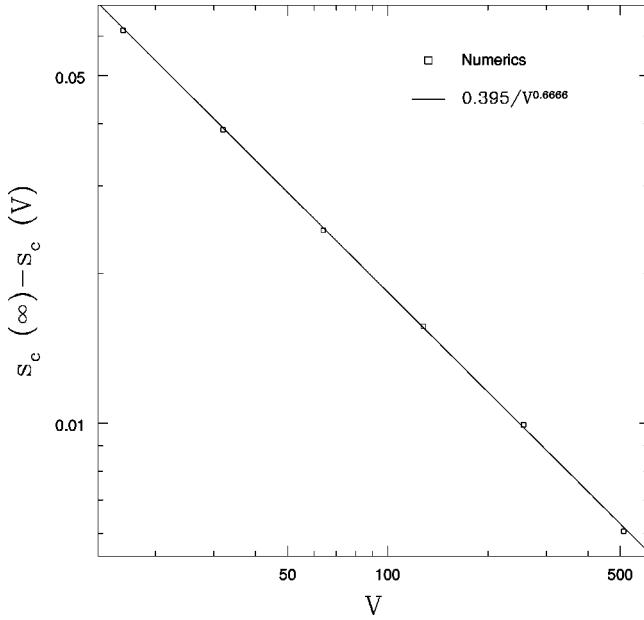
FIG. 3. The probability of a site being occupied, $s_c(V)$, converges to its asymptotic value, $s_c(\infty)$, as a power law, i.e., $s_c(\infty) - s_c(V) \sim 1/V^{0.66}$. The data are for lattices in one dimension.

off is replaced by 0, the integral on the left hand side of Eq. (22) is of $O(1)$. Comparing this with the right hand side of Eq. (22), we immediately arrive at the scaling relation

$$\phi(\tau - 1) = 1. \qquad (23)$$

If $\tau = 5/2$, Eq. (23) would indicate $\phi = 2/3$.

We found that the cleanest way to measure the exponent ϕ is by the following indirect finite size method. On a finite lattice, the critical value of the occupation probability of a site, $s_c(V)$, will differ from the $V = \infty$ value, $s_c(\infty)$. One can assume a reasonable finite size correction of the form

$$s_c(V) = s_c^{\infty} - \frac{a}{V^{\theta}}, \qquad (24)$$

where a is a constant and θ is a new exponent. Using this ansatz along with the expression for $\langle m^2 \rangle$ obtained from Eq. (20) in Eq. (7), we find $V^{\phi(3-\tau)-1} \sim V^{-\theta}$, implying another scaling relation,

$$\theta = 1 - \phi(3 - \tau). \qquad (25)$$

If $\tau = 5/2$ and $\phi = 2/3$, we get from Eq. (25) $\theta = 2/3$. In Fig. 3, we plot, in one dimension, $s_c(V)$ for various values of V and indeed find that $\theta = 2/3$. This provides an indirect numerical check on these scaling assumptions as well as suggesting, once again, that $\tau = 5/2$ is a superuniversal exponent.

Another indirect check can be done by mapping the one-dimensional CM to a fluctuating interface model [15]. We outline the procedure in brief. The first step is to map the CM onto a system of hard core particles moving on a ring. For this, interpret m_i, the mass at site i, to be the gap between the $(i-1)$th and ith particles. Then the chipping move corresponds to a particle jumping forward one step provided the

target site is empty, while the aggregation move corresponds to a particle making a long range jump to the site adjacent to the particle nearest to it. There is a standard procedure to map a lattice gas configuration to an interface configuration [26]. Let $n_j = 1 \ (-1)$ if a particle is present (absent) at site i. Then $h_i = \sum_{j=1}^{i} n_j$. While the chipping move corresponds to a local move of the interface, the aggregation move corresponds to a nonlocal move of the interface. The width of this interface was monitored numerically [15] as a function of time t. On the critical line $\rho_c(w)$ of the CM, the width was found to have the scaling form [15]

$$w(V, t) \approx V^{\chi} f\left(\frac{t}{V^z}\right), \qquad (26)$$

with $\chi \approx 0.67$ and $z \approx 2.0$.

We show here that the roughness exponent χ can be related to the exponents τ and ϕ of the CM. We first map the CM in 1D to an interface model with the height field $h_i = \sum_{j=1}^{i} m_j$. Then $\langle h_i \rangle = i\rho$ and $\langle h_i^2 \rangle \approx i\langle m^2 \rangle + i(i-1)\rho^2$. If we approximate $\langle m_i m_j \rangle$ by ρ^2 [assuming that the mean field $P(m)$ is exact], the width is then simply $w^2 = 1/V \sum_{i=1}^{V} (h_i - \langle h_i \rangle)^2 \sim V^{1+\phi(3-\tau)} = V^{2\chi}$. This implies that provided $\langle m_i m_j \rangle \approx \rho^2$ one gets the scaling relation

$$\chi = \frac{1}{2}[1 + \phi(3 - \tau)]. \qquad (27)$$

If $\tau = 5/2$ and $\phi = 2/3$, we get from Eq. (27) $\chi = 2/3$, in excellent agreement with the numerical value $\chi \approx 0.67$ [15]. This is further evidence for $\tau = 5/2$ and $\phi = 2/3$.

V. SUMMARY AND OUTLOOK

In this paper, we have studied a simple stochastic lattice model where masses diffuse as a whole and coalesce upon contact with rate 1, and every nonzero mass gives a single unit of mass to a neighbor with rate w. The mass density ρ is conserved by the dynamics. This model undergoes a nonequilibrium phase transition in the (ρ-w) plane across a critical line $\rho_c(w)$. We have shown analytically that $\rho_c(w) = \sqrt{w+1} - 1$ in all dimensions. We also provided both direct and indirect numerical evidence that strongly suggests that the mean field result for the single site mass distribution function $P(m)$ might be exact in all dimensions and that the exponent $\tau = 5/2$ is superuniversal.

However, we would like to stress one important point. Even though the single site distribution $P(m)$ may be given exactly by the mean field result, that does not prove that mean field theory or product measure is the exact stationary state in all dimensions. In this sense, the CM is different from the recently studied q model of force fluctuations [27], where the product measure is exact in the stationary state. More precisely, in the q model, the evolution equation for single site mass distribution $P(m)$ involves the two-point distribution $P(m_1, m_2)$. Similarly, the equation for the two-point involves the three-point function $P(m_1, m_2, m_3)$, and so on. However, if one assumes the product measure, i.e.,

$P(m_1, m_2, \ldots, m_n) = P(m_1)P(m_2) \cdots P(m_n)$ for all n, then it was shown [27] that this ansatz satisfies all the equations of the hierarchy. In the CM, in a similar way, one can write down the full hierarchy of equations satisfied by the n-point distribution functions. However, unlike in the q model, the product measure ansatz does not satisfy all the equations of this hierarchy [28]. Nevertheless, the expression for $P(m)$ obtained from the first equation of this hierarchy [the equation that involves only $P(m)$ and $P(m_1, m_2)$] seems to be extremely close to the numerical result. This suggests that in the stationary state of the CM correlations between masses appear only in the third- or higher order correlations but seem to be absent at the two-point level. This remarkable fact was also noticed recently in another aggregation model, namely, the asymmetric random average process with sequential updates [19,20], suggesting that such unusual correlations may be more generic and less exceptional than might appear.

In this paper we have studied the undirected CM. As mentioned in the Introduction, the directed CM also has a qualitatively similar phase transition in the steady state, although the associated critical exponents are entirely different from in the undirected one. Also, the phase boundary in the ρ-w plane of the directed CM was found be quite different from the mean field phase boundary [15]. A proper understanding of the directed model remains an outstanding challenging problem.

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